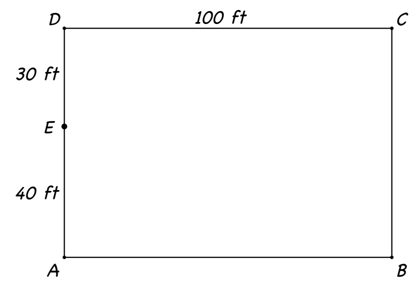
Ninth Annual Upper Peninsula
High School Math Challenge

Northern Michigan University (Marquette, MI, USA)
Saturday April 14, 2018

Team Problems — Solutions

1. The figure at the right represents a rectangular room with an electrical outlet located at E. If a point in the room is chosen at random, what is the probability that a 50-foot extension cord will reach that point?

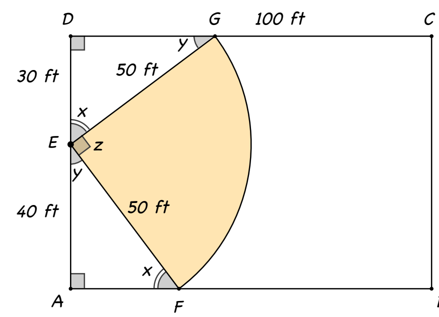


Answer: $\dfrac{48+25\pi}{280}$

$\angle A = \angle D = 90°$. By the Pythagorean theorem, both $\triangle AFE$ and $\triangle DEG$ are congruent 3–4–5 right triangles.

$\therefore$ $\angle AFE = \angle DEG = x$ and $\angle AEF = \angle DGE = y$ where $x + y = 90°$

$\therefore$ $\angle GEF = 90°$ and the shaded area is one-fourth of a circle of radius 50 ft.



Area of the room: $70 \times 100 = 7000$ ft^2

Area of $\triangle DEC$ = area of $\triangle AEF$ = $\dfrac{1}{2} \cdot 30 \cdot 40 = 600$ ft^2

Area of sector $= \dfrac{\pi \cdot 50^2}{4} = \dfrac{2500\,\pi}{4} = 625\,\pi$

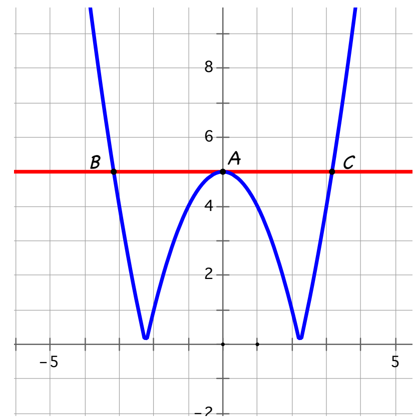
Accessible area $= 2 \cdot (600) + 625\,\pi$

Probability of connection $= \dfrac{1200 + 625\,\pi}{7000} = \dfrac{48 + 25\,\pi}{280}$

2. The graph of $g(x) = c$ intersects the graph of $f(x) = |x^2 - 5|$ at exactly three points. Find the numeric coordinates of the three points of intersection.

<u>**Answer:**</u> A $(0, 5)$, B $\left(-\sqrt{10}, 5\right)$, C $\left(\sqrt{10}, 5\right)$

The graph of $h(x) = \left(x^2 - 5\right)$ is a parabola that opens upward with vertex at $(0, -5)$. The graph of $f(x) = \left|x^2 - 5\right|$ reflects over the X-axis those points on $h(x)$ that lie below the X-axis. The vertex $(0, -5)$ is reflected to $(0, 5)$, as shown in the graph here.



$g(x) = c$ is a horizontal line. In order to intersect the graph of $f(x)$ in exactly three points, $c = 5$ and one intersection point is A $(0, 5)$. The other two points of intersection are at

$$f(x) = (x^2 - 5) = 5 \Rightarrow x^2 = 10 \Rightarrow x = \pm\sqrt{10}$$

The three intersections are: A $(0, 5)$, B $\left(-\sqrt{10}, 5\right)$, C $\left(\sqrt{10}, 5\right)$

3. If $(x + x^{-1}) = 3$, $(x^2 + x^{-2}) = a$, and $(x^3 + x^{-3}) = b$, find the numerical value of $a + b$.

<u>**Answer:**</u> $a + b = 25$

$$x + \frac{1}{x} = 3$$

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$$\left(x + \frac{1}{x}\right)^2 = 9$$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = \left(x^2 + \frac{1}{x^2}\right) + 2 = a + 2$$

$$a + 2 = 9 \Rightarrow a = 7$$

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$$\left(x + \frac{1}{x}\right)^3 = 27$$

$$x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \frac{1}{x^3} = 27$$

$$\left(x^3 + \frac{1}{x^3}\right) + 3\cdot\left(\frac{x^2}{x} + \frac{x}{x^2}\right) = \left(x^3 + \frac{1}{x^3}\right) + 3\cdot\left(x + \frac{1}{x}\right) = 27$$

$$b + (3\cdot3) = b + 9 = 27 \Rightarrow b = 18$$

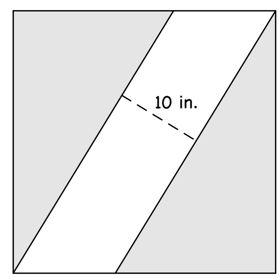
$$a + b = 7 + 18 = 25$$

4. How many perfect squares are divisors of the product $(1! \times 2! \times 3! \times \cdots \times 9!)$?

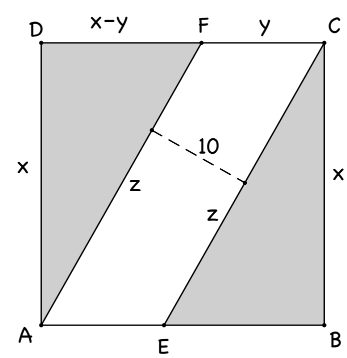
Answer: **672**

The product equals
$2\cdot(3\cdot2)\cdot(4\cdot3\cdot2)\cdot(5\cdot4\cdot3\cdot2)\cdot(6\cdot5\cdot4\cdot3\cdot2)\cdot(7\cdot6\cdot5\cdot4\cdot3\cdot2)\cdot(8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2)\cdot(9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2)$,
which equals $2^8\cdot3^7\cdot4^6\cdot5^5\cdot6^4\cdot7^3\cdot8^2\cdot9^1$,
which equals $2^8\cdot3^7\cdot2^{12}\cdot5^5\cdot2^4\cdot3^4\cdot7^3\cdot2^6\cdot3^2$,
and we have the prime factorization of the product: $2^{30}\cdot3^{13}\cdot5^5\cdot7^3$.
For a divisor of this number to be a perfect square, its prime factorization must be of the form $2^{\alpha}\cdot3^{\beta}\cdot5^{\gamma}\cdot7^{\delta}$, where α, β, γ, and δ are even; and
$0 \le \alpha \le 30$, $0 \le \beta \le 13$, $0 \le \gamma \le 5$, and $0 \le \delta \le 3$.
Sixteen choices are possible for α, seven choices are possible for β, three choices are possible for γ, and two choices are possible for δ. A total of
$16 \times 7 \times 3 \times 2$, or **672**, divisors of $1! \cdot 2! \cdot 3! \cdot \cdots \cdot 9!$ are perfect squares.

5. A square is divided into three pieces of equal area by making two parallel cuts, as shown. The distance between the parallel lines is 10 inches. What is the area of the square?



Answer: 1300 in^2



$$\text{Area } ABCD = x^2 \qquad \text{Area } AECF = \frac{x^2}{3} = 10\cdot z$$

$$\text{Area } \triangle AFD = \text{Area } \triangle EBC = \frac{x^2}{3} = \frac{x^2 - xy}{2}$$

$$\frac{x^2}{3} = \frac{x^2}{2} - \frac{xy}{2} \Rightarrow \frac{xy}{2} = \frac{x^2}{6} \Rightarrow xy = \frac{x^2}{3}$$

$$xy = \frac{x^2}{3} \Rightarrow y = \frac{x}{3} \Rightarrow (x-y) = \frac{2x}{3}$$

$$z^2 = x^2 + \left(\frac{2x}{3}\right)^2 = x^2 + \frac{4x^2}{9} = \frac{13x^2}{9}$$

$$z = \frac{x\sqrt{13}}{3} \quad \therefore \quad \frac{x^2}{3} = 10\cdot\frac{x\sqrt{13}}{3} \Rightarrow x = 10\sqrt{13}$$

$$\text{Area } ABCD = x^2 = 1300 \text{ in}^2$$